Investigation on the Paradox Raised by Gabriel's Horn
Candidate Code: jzt238

When I learned about improper integrals in math class, I thought that it was so neat that improper integrals defied boundaries because it allows us to find area/volume even when the solid reached infinity. While doing some homework problems, I ran into several solids that looked like a horn or trumpet, which led me to a mind-boggling concept known as the Painter's Paradox, which describes the characteristics of a solid known as Gabriel's Horn. It was named after the horn used by the Archangel Gabriel to announce the second coming of Jesus in the book of Revelations, and can also be referred to as Torricelli's Trumpet (Evangelista Torricelli). Gabriel's Horn is a solid that has a finite volume, but an infinite surface area, which when I think about it, does not really make sense. My aim in this investigation is to generate possible reasons as to why this is and why it also casts an infinite shadow. I will do this by solving for the volume, surface area, and area of the shadow casted by Gabriel's Horn using improper integrals and evaluating limits using L'Hopital, as well as explore the implications that the paradox of Gabriel's Horn raises on other similar solids.

What is Gabriel's Horn?

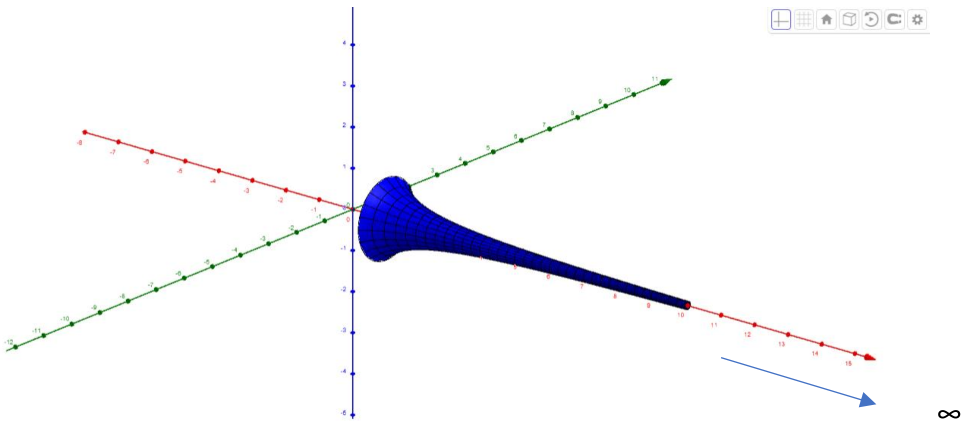


Figure 1 3D diagram of Gabriel's Horn using template on GeoGebra by Steven Phelps

Gabriel's Horn is generated by revolving the curve $\frac{1}{x}$, given that $x \geq 1$ around the x-axis from $x = 1$ to infinity. From the diagram above, I can see that as x approaches infinity, the "pipe-like" portion of the horn also keeps extending towards infinity, suggesting that the surface area of the horn is infinite. As x approaches infinity, the curve $\frac{1}{x}$ can also be seen approaching the x-axis, nearing 0, indicating that at the narrow end of the horn, the volume is finite.

Volume of Gabriel's Horn

To begin conceptualizing the Painter's Paradox, I will first show that Gabriel's Horn has a finite volume. Given the curve $y = \frac{1}{x}$, for $x \geq 1$: (cont.)

Graph of $\frac{1}{x}, x \geq 1$

Figure 2 Graph I created on Desmos

The solid is then generated by revolving the curve about the x-axis, from which we obtain a typical element of a disk. This can also be seen by reflecting the curve about the x-axis as seen below:

Graph of the revolution of the line $\frac{1}{x}$

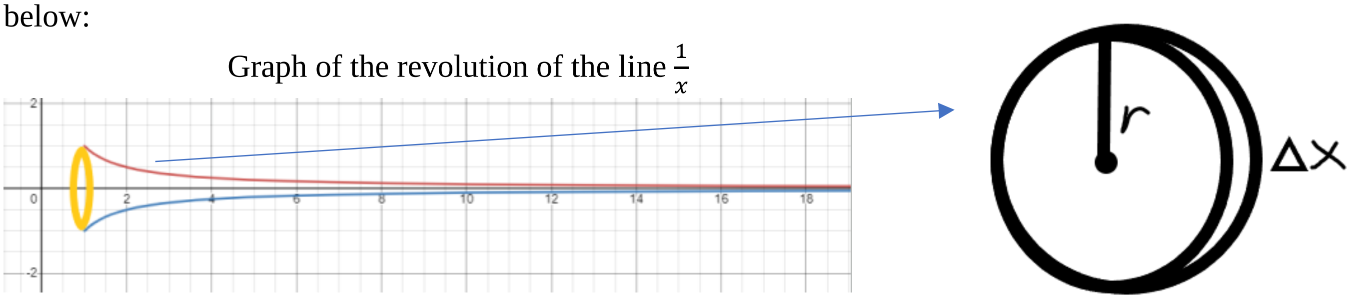


Figure 3 Diagram I created using Paint

Zooming in on the disk, we can see that the radius of the disk extends from the center point on the x-axis to the curve, so the radius is the height of the curve in terms of x ($r = \frac{1}{x}$). In addition, since the cross section of the disk is a circle with area πr^2, the volume is times the thickness (Δx). Therefore, by a using Riemann Sum to add up all the volumes of these disks in the revolution, we can obtain the exact value of the volume of the solid:

$$V = \pi r^2 \Delta x$$

$$\text{Given } r = \frac{1}{x}$$

$$V = \lim_{\Delta x \to 0} \sum_{k=1}^{n} \pi \left(\frac{1}{x}\right)^2 \Delta x$$

As x approaches 0, the sum of all the volumes of the disks approach the exact volume of the solid, which allows us to solve for the limit as a definite integral (an improper integral in my case, as the domain extends to infinity):

$$V = \pi \int_{1}^{\infty} \left(\frac{1}{x}\right)^2 dx$$

To evaluate this integral, I will replace the infinite limit with the constant b so that I can take the limit as b approaches infinity:

$$V = \lim_{b \to \infty} \int_{1}^{b} \pi x^{-2} dx$$

$$V = \lim_{b\to\infty} -\pi x^{-1} \Big|_{1}^{b}$$
$$V = \lim_{b\to\infty} \left[-\frac{\pi}{b}\right] - [-\pi] = 0 + \pi = \pi$$
As b approaches infinity, the limit is 0, so the volume is simply $0 + \pi$, which equals π. Therefore, the volume is finite.

Surface Area of Gabriel's Horn

To find the surface area of Gabriel's Horn, I first need to find the circumference of the base of each individual slice of the solid. Using the method of linear approximation, I will draw the slope of the tangent line for the curve $y = \frac{1}{x}$, for $x \geq 1$ to approximate the circumference:

Graph of the slope of the $\frac{1}{x}$

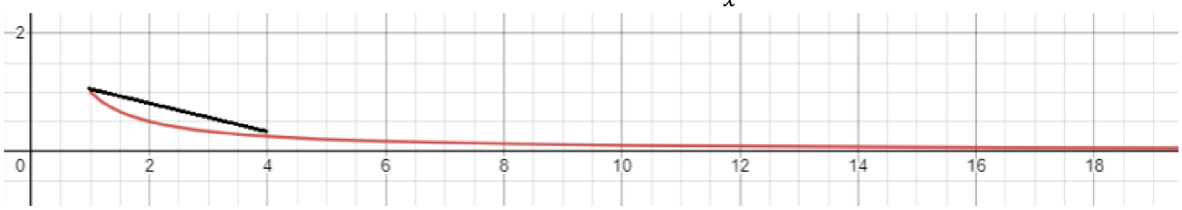


Figure 4 Diagram I created on Desmos

By rotating this line of approximation about the x-axis, I obtain a portion of a solid sliced from a cone known as a frustum:

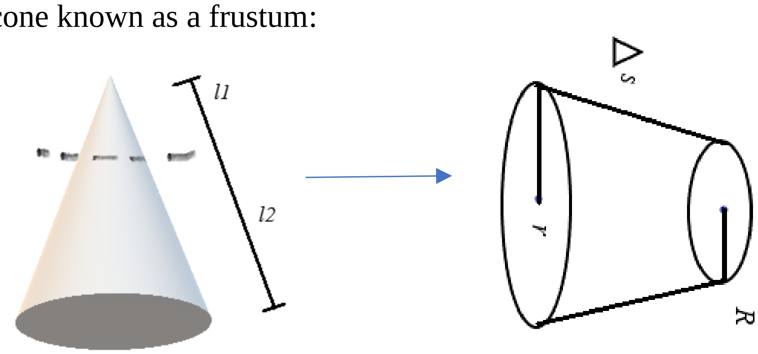


Figure 5 Image of a conical frustum I made on Paint

The surface area of a frustum is therefore the area of the top portion of the cone subtracted by the total area of the cone:
$$A = \pi r(l_1 + l_2) - \pi R l_1$$
$$A = \pi((r - R)l_1 + r l_2)$$
Using similar triangles:
$$\frac{l_1}{R} = \frac{l_1 + l_2}{r}(cont.)$$
$$r l_1 = R l_1 + R l_2$$
$$(r - R)l_1 = R l_2$$

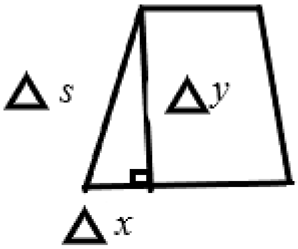
Therefore, after substituting everything back in the original equation, we obtain: (cont.)

$$A = \pi(Rl_1 + Rl_2 + rl_2 - Rl_1)$$
$$A = \pi(Rl_2 + rl_2)$$

This can be simplified to $A = 2\pi rl$[1], since $r = \frac{1}{2}(R + r)$.

To find Δs, which is the approximate length of the curve, we can use the Pythagorean Theorem to create a right triangle:

Figure 5 Diagram I created on Paint



$$\frac{(\Delta s)^2}{(\Delta x)^2} = \frac{(\Delta x)^2}{(\Delta x)^2} + \frac{(\Delta y)^2}{(\Delta x)^2}$$
$$(\frac{\Delta s}{\Delta x})^2 = 1 + (\frac{\Delta y}{\Delta x})^2$$
$$\frac{\Delta s}{\Delta x} = \sqrt{1 + (\frac{\Delta y}{\Delta x})^2}$$
$$\Delta s = \sqrt{1 + (\frac{\Delta y}{\Delta x})^2 \Delta x}$$

From here, we can use a Riemann Sum to arrive to the integral to solve for s.

$$\lim_{\Delta s \to 0} \sum_{n=0}^{\infty} \sqrt{1 + (\frac{\Delta y}{\Delta x})^2 \Delta x}$$

This is equivalent to: $= \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$

Since the radius of the frustum is equivalent to the distance from the x-axis to the curve, as can be seen below:

Graph of the revolution of $\frac{1}{x}$

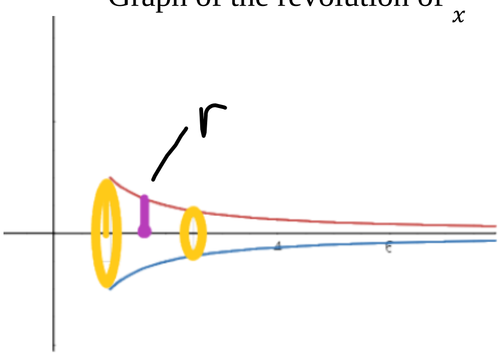


Figure 6 Diagram I created on Paint

With all the components, I can substitute this back into the surface area of a frustum formula:

$$A = 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx(cont.)$$

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[1] Paul Dawkins, Surface Area (Lamar University, 2003), https://tutorial.math.lamar.edu/classes/calcii/surfacearea.aspx

Using Riemann Sums to add up all the surface areas of the frustums:

$$A = \lim_{x\to\infty} \sum_{k=1}^{n} 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \left(\frac{dy}{dx}\right)^2 dx}$$

Thus, this gives us the integral to solve for the surface area of Gabriel’s Horn:

$$A = 2\pi \int_1^{\infty} \left(\frac{1}{x}\right) \sqrt{1 + \left(\frac{dy}{dx}\right)^2 dx}$$

Given $y = \frac{1}{x}, \frac{dy}{dx} = -\frac{1}{x^2}$

$$A = 2\pi \int_1^{\infty} \left(\frac{1}{x}\right) \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx$$

$$A = \lim_{b\to\infty} 2\pi \int_1^{b} \left(\frac{1}{x}\right) \sqrt{\frac{x^4+1}{x^4}}\, dx(cont.)$$

$$A = \lim_{b\to\infty} 2\pi \int_1^{b} \left(\frac{\sqrt{x^4+1}}{x}\right) dx$$

To approach this, I will focus on solving the indefinite integral:

$$\int \frac{\sqrt{x^4+1}}{x} dx$$

Using double u-substitution:

$$Let\ u = \sqrt{x^4+1}$$

$$du = \frac{1}{2\sqrt{x^4+1}} \times 4x^3 dx$$

$$du = \frac{2x^3}{\sqrt{x^4+1}} dx$$

$$\frac{\sqrt{x^4+1}}{2x^3} du = dx$$

$$If\ u = \sqrt{x^4+1},\ then\ solving\ for\ x\ results\ in:$$

$$u^2 = x^4 + 1$$

$$x = \pm\sqrt[4]{u^2-1}\ (consider\ the\ positive\ solution\ only\ because\ of\ domain)$$

Replacing integral with du:

$$\int \frac{u^2}{2(\sqrt[4]{u^2-1})^4} du$$

$$= \frac{1}{2} \int \frac{u^2}{u^2-1} du$$

Using a carefully chosen 0:

$$= \int \frac{(u^2-1)+1}{u^2-1} du$$

$$= \int \frac{u^2-1}{u^2-1} + \frac{1}{u^2-1} du$$ [2]

$$= \int 1 + \frac{1}{u^2-1} du$$

Using the method of partial fractions to solve $\int \frac{1}{u^2-1}\ du$:

$$\frac{1}{u^2-1} = \frac{A}{(u-1)} + \frac{B}{(u+1)}$$

$$1 = A(u+1) + B(u-1)$$

$$1 = (A+B)u + (A-B)$$

Setting equal like coefficients:

$$0 = A + B$$

$$1 = A - B$$

$$\overline{}$$

$$B = -\frac{1}{2}, A = \frac{1}{2}$$

$$\therefore \int 1 + \frac{1}{2} \cdot \frac{1}{(u-1)} + -\frac{1}{2} \cdot \frac{1}{(u+1)} du$$

$$= u + \frac{1}{2} ln|u-1| - \frac{1}{2} ln|u+1| + c(cont.)$$

Substituting back in for u:

$$\sqrt{x^4+1} + \frac{1}{2} ln\left|(\sqrt{x^4+1}) - 1\right| - \frac{1}{2} ln\left|\left(\sqrt{x^4+1}\right) + 1\right| + c$$

Now that I solved the integral, I can substitute it back into the integral that solves for the surface area:

$$A = \lim_{b\to\infty} 2\pi[\sqrt{x^4+1} + \frac{1}{2} ln\left|\left(\sqrt{x^4+1}\right) - 1\right| - \frac{1}{2} ln\left|\left(\sqrt{x^4+1}\right) + 1\right|] \Big|_1^b$$

$$A = \lim_{b\to\infty} 2\pi[\sqrt{x^4+1} + \frac{1}{2} \cdot ln \frac{\left|\left(\sqrt{x^4+1}\right) - 1\right|}{\left|\left(\sqrt{x^4+1}\right) + 1\right|}] \Big|_1^b$$

$$A = \lim_{b\to\infty} 2\pi \left[\sqrt{b^4+1} + \frac{1}{2} \cdot ln \frac{\left|\left(\sqrt{b^4+1}\right) - 1\right|}{\left|\left(\sqrt{b^4+1}\right) + 1\right|} - \left(1 + 0 + 1 - (0-1)\right)\right] = \infty + 3 = \infty$$

[2] Trevor L.A., Properties, Operations, Tricks of Fractions, https://www.csusm.edu/mathlab/documents/properties-of-fractions.pdf

When solving for the limit as b approaches infinity, anything added by ∞, which is what the limit as b approaches ∞ is for $\sqrt{b^4+1}$. As b approaches infinity for $ln\frac{|(\sqrt{b^4+1})-1|}{|(\sqrt{b^4+1})+1|}$, the solution is also indeterminate. Therefore, the solution to the improper integral diverges.

$$\therefore Gabriel's\ Horn\ has\ infinite\ surface\ area.$$

Area of Shadow Casted by Gabriel's Horn

Since, Gabriel's Horn possesses infinite surface area, does that also mean that the area of the its shadow is also infinite? To determine if this is true, I will the area of the projection of Gabriel's Horn on the xy-plane, taking into consideration that the solid is bounded by the curve $y = \frac{1}{x}$, for $x \leq 1$ and the x-axis.

Using integration with the same limits as before:

$$A = \int_1^\infty \frac{1}{x} dx$$

$$A = \lim_{b\to\infty} \int_1^b \frac{1}{x} dx\ (cont.)$$

$$A = \lim_{b\to\infty} ln|x| \Big|_1^b$$

$$A = \ln(\mathrm{b}) - 0 = \infty$$

With the area of the shadow casted by Gabriel's Horn being infinite, this affirms that the area of Gabriel's Horn is infinite in 2-dimension, as well as 3-dimension.

What do the results from Gabriel's Horn suggest about the volume and surface area of other similar solids?

Consider the form: $f(x) = \frac{1}{x^a}$, which is in the form of a P-integral. In the case of Gabriel's Horn, $a = 1$, which generates a solid with finite volume and infinite area. Here, I will explore what other cases possess the same behavior as Gabriel's Horn. \

For $a < 1$:

Consider: $f(x) = \frac{1}{x^{\frac{1}{2}}}$

Using the formula for volume of a solid of revolution:

$$V = \pi \int_1^\infty \left(\frac{1}{x^{\frac{1}{2}}}\right)^2 dx$$

$$V = \lim_{x\to\infty} \pi \int_1^b x^{(-\frac{1}{2})} dx\ (cont.)$$

$$V = \lim_{x\to\infty} 2x^{\frac{1}{2}} \Big|_{1}^{b}$$

$$V = \infty - 2 = \infty$$

Using the formula for surface area of a solid of revolution:

$$A = 2\pi \int_{1}^{\infty} (x^{\left(-\frac{1}{2}\right)})\sqrt{1 + \left(f'(x)\right)^2}dx$$

$$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$A = 2\pi \int_{1}^{\infty} (x^{\left(-\frac{1}{2}\right)})\sqrt{1 + \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)^2}\, dx$$

Using my GDC, I can see that the surface area converges to ≈381.68.

Therefore, for cases where $a < 1$, the convergence/divergence behavior the reverse of that of Gabriel's Horn.

For $a > 1$:

Consider: $f(x) = \frac{1}{x^2}$.

Using the formula for volume of a solid of revolution:

$$V = \pi \int_{1}^{\infty} (\frac{1}{x^2})^2\, dx(cont.)$$

$$V = \lim_{x\to\infty} \pi \int_{1}^{b} \frac{1}{x^4} dx$$

$$V = \lim_{x\to\infty} -\frac{1}{3}x^{-3} \Big|_{1}^{b}$$

$$V = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$$

Using the formula for surface area of a solid of revolution:

$$A = 2\pi \int_{1}^{\infty} (x^2)\sqrt{1 + \left(f'(x)\right)^2}dx$$

$$f'(x) = 2x\ (cont.)$$

$$A = 2\pi \int_{1}^{\infty} (x^2)\sqrt{1 + (2x)^2}\, dx$$

$$A = 2\pi \int_{1}^{\infty} (x^2)\sqrt{1 + 4x^2}dx$$

Using my GDC, I can see that the surface area gradually diverges. Also, solely examining the length of curve confirms this because as x approaches infinity, the length also approaches infinity.
Therefore, for cases where $a > 1$, the convergence/divergence behavior is the same as that of Gabriel's Horn.

Possible reasons for these patterns:
For $a < 1$, the solids of revolution produced have an infinite volume because the disk at any point on x does not decrease towards 0 fast enough to converge. For $a > 1$, the solids have a finite volume because the disk decreases faster to 0. The same logic can be applied for the surface area in terms of whether the length of the curve is approaching infinity faster. For $a < 1$, the length of the curve rapidly converges to a value faster, while for $a > 1$ is slower and thus will eventually diverge to infinity.

Conclusion
Having demonstrated that the solid possess both finite volume and infinite surface area, I want to explain how this could be possible by examining the context around which the Painter's Paradox was raised. For me, it is easy to conceptualize why the painter cannot paint the surface of Gabriel's Horn because as the curve approaches the x-axis, as x approaches infinity, the length of the horn also extends to infinity. If a painter were to attempt to coat its surface, they would have trouble coating the narrow end of the horn, as that is where the surface areas of the slices get infinitely smaller. It also extends on forever, so the painter would have to spend an eternity painting it, which is impossible. The same reasoning extends for the area of the shadow casted by Gabriel's Horn because as the length keeps heading towards infinity so will the length of the shadow. Additionally, the Painter's Paradox raises another interesting idea that it is possible to coat the interior of Gabriel's Horn by pouring in a finite amount of paint, as the volume is finite. However, Gabriel's Horn cannot hold the amount of paint poured in, even though it has finite volume. This seems to suggest that the horn has a hole at the narrow end. It is understood that while solids with a hole can still have a finite volume (as the volume can be obtained by subtracting the total volume by the volume of the hole), there is a different type of hole in Gabriel's Horn. At the narrow "pipe-like" end, it can be argued that there is an extremely thin "hole" that is generalized to be negligible in the volume calculation because as x approaches infinity, the curve approaches the x-axis, almost nearing 0. This solution to the limit allows for Gabriel's Horn to have a finite volume with zero capacity. While, the rules of calculus work in solving for the volume and surface area of Gabriel's Horn, it raises questions as to what extent can improper integrals and infinite limits model the behavior of real-life solids if it creates paradoxes within general calculus? Intuitively, a solid with infinite surface area should also have infinite volume, but when I was solving for both, this was not the case, as integrals for volume and surface area lead to different behaviors of convergence or divergence, not just for Gabriel's Horn, but for other similar solids as well. I think that this could imply that

we need to further inspect the nature of improper integrals and find a more accurate way to approximate the volumes and surface areas of solids with an infinite limit. A possible extension of my investigation, would be to explore the underlying concept of improper integrals, which are Riemann Sums and determine which Riemann Sums (LRAM, RRAM, Middle, Trapezoidal) is most accurate for solving for the volume and surface area of infinite solids.

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